TuT 6

Consider $\vec{x}' = \vec{P}(t)\vec{x}$, and let $x^{(1)}(t) \dots x^{(n)}(t)$ be the linearly independent solution of $\vec{x}' = \vec{P}\vec{x}$

then $\psi(t) = \left(\vec{x}^{(1)}(t) \; \dots \; \vec{x}^{(n)}(t)\right)$ is the fundamental matrix.

The general solution of $x' = Px$ is $\vec{x} = \psi(t)\vec{c}$

If we have an initial condition $\vec{x}(t_0) = \vec{x}^0$,

then $\psi(t_0)\vec{c} = \vec{x}^0 \Rightarrow \vec{c} = (\psi(t_0))^{-1}\vec{x}^0$

$$\therefore \vec{x} = \psi(t) \cdot (\psi(t_0))^{-1}\vec{x}^0$$

then $\phi(t) = \psi(t)(\psi^{-1})(t_0)$ is the fundamental matrix with $\phi(t_0) = I$.

Diagonalize a Matrix.

Suppose $A \in M^{n \times n}$ has n linearly independent eigenvectors $\vec{\xi}^1, \vec{\xi}^2 \dots \vec{\xi}^n$, then A can be diagonalized by $T^{-1}AT = D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$

where $T = \left(\vec{\xi}^1 \; \vec{\xi}^2 \; \dots \; \vec{\xi}^n\right)$ and λ_i are eigenvalues.

If we have a Nonhomogeneous Linear system $\vec{X}' = \vec{A}\vec{X} + \vec{g}$,

there are 3 method to compute the solution

① Diagonalization

$$\vec{X}' = A\vec{X} + g(t)$$

let T be the matrix whose columns are eigenvectors of A.

let $X = Ty$

$$Ty' = ATy + g(t)$$

$$y' = T^{-1}ATy + T^{-1}g(t)$$

$$= Dy + T^{-1}g(t)$$

where $D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$ λ_i are eigenvalues.

then $\begin{cases} y_1' = \lambda_1 y_1 + \hat{g}_1(t) \\ \vdots \\ y_n' = \lambda_n y_n + \hat{g}_n(t). \end{cases}$

② Undetermined coefficient. (not recommanded

If $g(t)$ is in form of $e^{\lambda t}$, then

we should guess $\vec{x} = \vec{a}te^{\lambda t} + \vec{b}e^{\lambda t}$ (not only $\vec{a}te^{\lambda t}$

Similarly if we have another form of $g(t)$.

③ Variation of Parameters

Suppose $\vec{x}' = P(t)\vec{x} + \vec{g}(t)$, then

let $\Psi(t)$ is the fundamental matrix corresponding

to $\vec{x}' = P(t)\vec{x}$, then

$$\vec{x} = \Psi(t)\left(\Psi^{-1}(t_0)\right)\vec{x^0} + \Psi(t)\int_{t_0}^{t} \Psi^{-1}(s)\, g(s)\, ds.$$

Problem:

①
$$x' = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix} x + \begin{pmatrix} \csc t \\ \sec t \end{pmatrix}$$

Ans:
$$\begin{vmatrix} 2-t & -5 \\ 1 & -2-t \end{vmatrix} = 0$$

$$(t-2)(t+2)+5 = 0$$

$$t = \pm i$$

$$\vec{\xi}_1 = \begin{pmatrix} 2+i \\ 1 \end{pmatrix} \text{ corr. to } \lambda_1 = i.$$

So $\begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it}$

$$= \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + i \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

$$\vec{x} = c_1 \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix} + c_2 \begin{pmatrix} \cos t + 2\sin t \\ \sin t \end{pmatrix}$$

fundamental matrix is

$$\psi = \begin{pmatrix} 2\cos t - \sin t & \cos t + 2\sin t \\ \cos t & \sin t \end{pmatrix}$$

So $\psi^{-1} = \begin{pmatrix} -\sin t & \cos t + 2\sin t \\ \cos t & -2\cos t + \sin t \end{pmatrix}$

$$\vec{x} = \vec{\psi}\vec{c} + \vec{\psi} \int_0^t \psi^{-1} g$$

$$\int_0^t \psi^{-1} g = \int_0^t \begin{pmatrix} -\sin s & \cos s + 2\sin s \\ \cos s & -2\cos s + \sin s \end{pmatrix} \begin{pmatrix} \csc s \\ \sec s \end{pmatrix}$$

$$= \int_0^t \begin{pmatrix} 2\tan s \\ \cot s - 2 + \tan s \end{pmatrix}$$

$$= \begin{pmatrix} -2\ln\cos t \\ -2t + \ln\tan t \end{pmatrix}$$

So $\vec{x} = \vec{\psi}\vec{c} + \vec{\psi} \begin{pmatrix} -2\ln\cos t \\ -2t + \ln\tan t \end{pmatrix}$